UDC: 519.833.2

MSC2010: 91A06

BERGE-VAISMAN EQUILIBRIUM FOR ONE LINEAR-QUADRATIC DIFFERENTIAL GAME

© V. I. Zhukovskiy

Moscow State University named after Lomonosov
Faculty of Computational Mathematics and Cybernetics
Department of Optimal Control
Leninskiye Gory, GSP-1, Moscow, 119991, Russian Federation
e-mail: *zhkvlad@yandex.ru*

© L. V. Smirnova

Moscow State University of Technologies and Management named after Rasumovskiy
Zemlyanoy Val, 73, Moscow, 109004, Russian Federation
e-mail: *smirnovalidiya@rambler.ru*

Berge-Vaisman equilibrium for one linear-quadratic differential game.

Zhukovskiy V. I., Smirnova L. V.

Abstract. We obtained coefficient criteria for the existence of the Berge-Vaisman equilibrium in a non-cooperative positional linear-quadratic game of two persons with small parameter.

Keywords: *non-cooperative positional linear-quadratic differential game, dynamic programming, Berge-Vaisman equilibrium, Nash equilibrium, continuous dependence and analyticity of the solution by parameter*

Introduction

The notion of "Berge equilibrium" appeared in Russia in 1994 during the critical discussion of the published book [1] by Claude Berge in Paris. Berge equilibrium (BE) removes "selfish" nature promoted in [1] Nash equilibrium due to the altruistic approach, dictated by the concept of BE. In 1995 Constantin S. Vaisman (then a graduate student of Zhukovskiy) defended his Ph.D. thesis on Berge equilibrium, at the Leningrad University in 1995. Unfortunately, Vaisman died three years after the Ph.D. defense of the thesis, before the age of 36 years. During these three years he published 19 works, a list of which we present at the end of the article. We observe also that Vaisman together with the first author of this article wrote individual chapters in two books [9, 19].

Vaisman merit lies in the fact that he presented the example that the property of individual rationality for BE, generally speaking, does not take place. Therefore Vaisman added this requirement in the definition of Berge equilibrium, after which BE was naturally called as the equilibrium by Berge–Vaisman.

BE has not got the bright destiny. Because of Vaisman's death, who was the greatest enthusiast of Berge equilibrium, they suspended the investigation of it in Russia. Furthermore, the publication of the book [1] aroused the acute review of Martin Shubic. However, the Algerian trainees of Zhukovskiy Radjef Mohamed Said and Larbani Moussa managed to publish the works [21, 22], which caused widespread interest in the West to BE. As shown by the review [23], right now the research of BE stuck at an early stage. Namely, they are the initial accumulation of facts, the formalization of modification BE, a comparison with Nash equilibrium. Futhermore, basically, all the studies are limited to only finite non-cooperative games. We believe it is time to proceed to the second heuristic stage, that is to answer the following two questions:

1) Is there Berge equilibrium and how to build it?

2) How should one take into account the dynamics of the conflict?

The recently published book [24] was dedicated to the answer to the first question, it is true only within static version of non-cooperative games. We expect to dedicate a separate book to Dynamic version of the problem (within the mathematical formalization of the positional differential game proposed by Russian academician Krasovsky). Though this article is devoted to a partial answer to the second question.

1. Formulation of the problem

Let us consider a non-cooperative positional linear-quadratic differential game of two persons, where one of the players has small influence on the rate of the change of the phase vector

$$\langle \{1,2\}, \Sigma, \{\mathfrak{A}_i\}_{i=1,2}, \{\mathscr{J}_i(U,t_0,x_0)\}_{i=1,2}\rangle. \quad (1)$$

Here $\{1,2\}$ is a set of the serial numbers of the players. Variation of the control system Σ with respect to the time t is described by the linear equation

$$\dot{x} = A(t)x + u_1 + \varepsilon u_2, \quad x(t_0) = x_0,$$

where the $n\times n$-dimensional matrix $A(\cdot) \in C_{n\times n}[t_0,\vartheta]$; $t_0 \geq 0$ is the moment of beginning and $\vartheta > t_0$ is the fixed moment of finishing of the game; $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$ is a phase n-column-vector. The pair $(t,x) \in [t_0,\vartheta]\times\mathbb{R}^n$ forms a position of the game, (t_0,x_0) is the initial position of the game, $(t,x(t))$ is the current position of the game at the moment of the time $t \in [t_0,\vartheta]$; $u_i \in \mathbb{R}^n$ is a control action of the i-th player ($i = 1,2$), the constant $\varepsilon \geq 0$ is the small scalar parameter. A set of strategies of the i-th player is $\mathfrak{A}_i = \{U_i \div u_i(t,x) = Q_i(t)x\}$, i. e., a strategy U_i of the i-th player identified with a vector-valued function $u_i(t,x)$ (denoted by $U_i \div u_i(t,x)$), that is linear with respect to $x \in \mathbb{R}^n$ and continuous with respect to t. Thus, the choice of strategy U_i for the i-th player means the choice of an $n \times n$-dimensional matrix $Q(\cdot) \in C_{n\times n}[0,\vartheta)$ with the continuous elements on interval $[0,\vartheta)$. A system $U = (U_1, U_2) \in \mathfrak{A} = \mathfrak{A}_1 \times \mathfrak{A}_2$ is called a situation of the game (1).

The game (1) proceeds in the following way. Each i-th player chooses its strategy $U_i \div u_i(t,x) = Q_i(t)x$, $U_i \in \mathfrak{A}_i$ $(i = 1,2)$. Substituting $u_i = u_i(t,x)$ in (1), we have an inhomogeneous linear system of differential equations containing continuous coefficient with respect to t:

$$\dot{x} = [A(t) + Q_1(t) + \varepsilon Q_2(t)]x, \quad x(t_0) = x_0.$$

For any fixed $\varepsilon = const \geq 0$ this system has the unique continuous solution $x(t)$, which can be extended for interval $[t_0,\vartheta]$. Using $x(\cdot) \in C_n[t_0,\vartheta]$, we construct *realizations* $u_i[t] = Q_i(t)x(t)$ of the chosen by the players strategies $U_i \in \mathfrak{A}_i$. On continuous triples $(x(t), u_1[t], u_2[t] \mid t \in [t_0,\vartheta])$ we define the payoff functions of the players, which are given by quadratic functionals:

$$\mathscr{J}_1(U,t_0,x_0) = x'(\vartheta)C_1x(\vartheta) + \int\limits_{t_0}^{\vartheta}(u_1'[t]D_{11}u_1[t] - u_2'[t]D_{12}u_2[t] + x'(t)G_1x(t))dt,$$

$$\mathscr{J}_2(U,t_0,x_0) = x'(\vartheta)C_2x(\vartheta) + \int\limits_{t_0}^{\vartheta}(-u_1'[t]D_{21}u_1[t] + u_2'[t]D_{22}u_2[t] + x'(t)G_2x(t))dt,$$

where the prime on the top indicates the operation of transposition, $n \times n$-dimensional matrices C_i, G_i, D_{ij} $(i,j = 1,2)$ are symmetric and constant. Further, the fact that the quadratic form $u'Mu$ is definitely negative (positive, nonnegative and nonpositive) we denote by $M < 0$ $(>, \geq, \leq)$.

Definition. A situation $U^B = (U_1^B, U_2^B) \in \mathfrak{A}$ is called a Berge-Vaisman equilibrium for the game (1) if the following conditions for any choice of the initial position $(t_0,x_0) \in [0,\vartheta) \times \mathbb{R}^n$ hold:

$$\begin{aligned} &\mathscr{J}_1(U_1^B, U_2, t_0, x_0) \leq \mathscr{J}_1(U^B, t_0, x_0) \ \forall U_2 \in \mathfrak{A}_2, \\ &\mathscr{J}_2(U_1, U_2^B, t_0, x_0) \leq \mathscr{J}_2(U^B, t_0, x_0) \ \forall U_1 \in \mathfrak{A}_1, \end{aligned} \qquad (2)$$

$$\max_{U_1}\min_{U_2} \mathscr{I}_1(U_1,U_2,t_0,x_0) \leq \mathscr{I}_1(U^B,t_0,x_0),$$
$$\max_{U_2}\min_{U_1} \mathscr{I}_2(U_1,U_2,t_0,x_0) \leq \mathscr{I}_2(U^B,t_0,x_0). \qquad (3)$$

2. Existence of the Berge-Vaisman equilibrium

So, we will consider a differential positional linear-quadratic game of two persons (1). Recall that dynamics of the game is described by the ordinary linear differential equation

$$\dot{x} = A(t)x + u_1 + \varepsilon u_2, \quad x(t_0) = x_0,$$

where $x, u_i \in \mathbb{R}^n$ $A(\cdot) \in C_{n\times n}[t_0,\vartheta]$, $\varepsilon = const$ is a small parameter, constant $\vartheta > t_0 \geq 0$. A set of strategies of the i-th player is

$$\mathfrak{A}_i = \{U_i \div u_i(t,x) = Q_i(t)x \mid \forall t \in [0,\vartheta), x \in \mathbb{R}^n, Q(\cdot) \in C_{n\times n}[0,\vartheta)\} \quad (i=1,2),$$

payoff functions of the players are:

$$\mathscr{I}_1(U_1,U_2,t_0,x_0) = x'(\vartheta)C_1x(\vartheta) + \int\limits_{t_0}^{\vartheta}(u_1'[t]D_{11}u_1[t] - u_2'[t]D_{12}u_2[t] + x'(t)G_1x(t))dt,$$

$$\mathscr{I}_2(U_1,U_2,t_0,x_0) = x'(\vartheta)C_2x(\vartheta) + \int\limits_{t_0}^{\vartheta}(-u_1'[t]D_{21}u_1[t] + u_2'[t]D_{22}u_2[t] + x'(t)G_2x(t))dt,$$

where symmetric constant matrices C_i, G_i, D_{ij} are such that $C_2 \leq 0$, $G_2 \leq 0$, $D_{ij} > 0$ $(i,j=1,2)$. Berge-Vaisman equilibrium is formalized by definition 1.

Statement. *If the constant $\varepsilon \geq 0$ is sufficiently small and the matrices*

$$C_2 \leq 0, G_2 \leq 0, D_{ij} > 0 \ (i,j=1,2), \qquad (4)$$

then a situation of Nash equilibrium doesn't exist, but a situation of Berge-Vaisman equilibrium

$$(U_1^B, U_2^B) \div (u_1^B(t,x), u_2^B(t,x)), \ \ U_i^B \in \mathfrak{A}_i \ \ (i=1,2)$$

exists in the game (1) for any choice of the initial position $(t_0,x_0) \in [0,\vartheta)\times\mathbb{R}^n$. Moreover, $u_i^B(t,x)$ can be represented in the form $u_i^B(t,x) = Q_i^B(t,\varepsilon)x$ $(i=1,2)$, where the matrices $Q_i^B(t,\varepsilon) \in C_{n\times n}[t_0,\vartheta]$ for mentioned constant $\varepsilon \geq 0$.

Proof. First of all we note two facts. Firstly, according to $D_{11} > 0$ and (or) $D_{22} > 0$ the situation of Nash equilibrium doesn't exist in the game (1) [25, p. 115]. Secondly, maximins in the left parts of the inequalities (3) don't exist too (by $D_{12} > 0$, $D_{21} > 0$ and [26, p. 272-273]). Therefore, to prove the existence of Berge equilibrium it is sufficient to establish the correctness of the implication $[D_{12} > 0, D_{21} > 0] \Rightarrow [\exists\ \forall (t_0, x_0) \in [0, \vartheta) \times \mathbb{R}^n]$. We can establish this fact using the method of the dynamic programming and Poincare's theory about small parameter [27]. According to the method of the dynamic programming we construct two scalar functions

$$\begin{aligned} W_1(t, x, u_1, u_2, V_1, V_2) &= \frac{\partial V_1}{\partial t} + \left[\frac{\partial V_1}{\partial x}\right]' [A(t)x + u_1 + \varepsilon u_2] + \\ &+ u_1' D_{11} u_1 - u_2' D_{12} u_2 + x' G_1 x, \\ W_2(t, x, u_1, u_2, V_1, V_2) &= \frac{\partial V_2}{\partial t} + \left[\frac{\partial V_2}{\partial x}\right]' [A(t)x + u_1 + \varepsilon u_2] - \\ &- u_1' D_{21} u_1 + u_2' D_{22} u_2 + x' G_2 x. \end{aligned} \tag{5}$$

In the usual way (such as in [25, p. 62-67]) the correctness of the next proposition can be established, where $V = (V_1, V_2) \in \mathbb{R}^2$, 0_n is the n-dimensional zero vector; $0_{n\times n}$ is the $n \times n$-dimensional zero matrix; $Idem\{u \to u^*\}$ implies that in the expression which is situated in the braces u is replaced by u^*:

Assume that for the game (1) we find two continuously differentiable scalar functions $V_i(t, x)$ $(i = 1, 2)$ such that the following conditions hold:

1. $$V_i(\vartheta, x) = x' C_i x \quad \forall x \in \mathbb{R}^n; \tag{6}$$
2. there are two n-dimensional vector-valued functions $u_i(t, x, V)$ $(i = 1, 2)$ such that following expressions
$$\begin{aligned} &\max_{u_2} \{W_1(t, x, u_1(t, x, V), u_2, V)\} = Idem\{u_2 \to u_2(t, x, V)\}, \\ &\max_{u_1} \{W_2(t, x, u_1, u_2(t, x, V), V)\} = Idem\{u_1 \to u_1(t, x, V)\} \end{aligned} \tag{7}$$
are valid for any $t \in [0, \vartheta)$, $x \in \mathbb{R}^n$, $V \in \mathbb{R}^2$;
3. there are the continuously differentiable solutions $V_i(t, x)$ $(i = 1, 2)$ of the system of two partial differential equation
$$W_i[t, x, V] = W_i(t, x, u_1(t, x, V), u_2(t, x, V), V(t, x)) = 0, \quad (i = 1, 2) \tag{8}$$
with boundary conditions (6);
4. the inclusions $U_i^B \div u_i^B(t, x) = u_i(t, x, V_1(t, x), V_2(t, x)) \in \mathfrak{A}_i$ $(i = 1, 2)$ hold.

Then:

a) situation Berge-Vaisman equilibrium has a form: $U^B = (U_1^B, U_2^B)$;

b) the payoff of the players are $\mathscr{J}_i(U^B, t_0, x_0) = V_i(t_0, x_0)$ $(i = 1, 2)$ for the any initial position $(t_0, x_0) \in [0, \vartheta) \times \mathbb{R}^n$.

Now, we use this proposition. Firstly, the requirements (7) are valid, if

$$\begin{aligned}
\left.\frac{\partial W_1(t,x,u_1(t,x,V),u_2,V)}{\partial u_2}\right|_{u_2=u_2(t,x,V)} &= \varepsilon\frac{\partial V_1}{\partial x} - 2D_{12}u_2(t,x,V) = 0_n,\\
\left.\frac{\partial W_2(t,x,u_1,u_2(t,x,V),V)}{\partial u_1}\right|_{u_1=u_1(t,x,V)} &= \frac{\partial V_2}{\partial x} - 2D_{21}u_1(t,x,V) = 0_n,
\end{aligned} \tag{9}$$

$$\frac{\partial^2 W_1(t,x,u_1(t,x,V),u_2,V)}{\partial u_2^2} = -2D_{12} < 0,$$

$$\frac{\partial^2 W_2(t,x,u_1,u_2(t,x,V),V)}{\partial u_1^2} = -2D_{21} < 0.$$

From (9) we get

$$\begin{aligned}
u_2(t,x,V) &= \frac{\varepsilon}{2}D_{12}^{-1}\frac{\partial V_1}{\partial x},\\
u_1(t,x,V) &= \frac{1}{2}D_{21}^{-1}\frac{\partial V_2}{\partial x}.
\end{aligned} \tag{10}$$

Taking into account (10) and (8) from (5) we have

$$\begin{aligned}
&W_1[t,x,V] = \frac{\partial V_1}{\partial t} + \left[\frac{\partial V_1}{\partial x}\right]' A(t)x + \frac{1}{2}\left(\frac{\partial V_1}{\partial x}\right)' D_{21}^{-1}\frac{\partial V_2}{\partial x} +\\
&+ \frac{\varepsilon^2}{4}\left(\frac{\partial V_1}{\partial x}\right)' D_{12}^{-1}\frac{\partial V_1}{\partial x} + \frac{1}{4}\left(\frac{\partial V_2}{\partial x}\right)' D_{21}^{-1}D_{11}D_{21}^{-1}\frac{\partial V_2}{\partial x} +\\
&+ x'G_1x = 0,\ V_1(\vartheta,x) = x'C_1x,\\
&W_2[t,x,V] = \frac{\partial V_2}{\partial t} + \left[\frac{\partial V_2}{\partial x}\right]' A(t)x + \frac{1}{4}\left(\frac{\partial V_2}{\partial x}\right)' D_{21}^{-1}\frac{\partial V_2}{\partial x} +\\
&+ \frac{\varepsilon^2}{2}\left[\frac{\partial V_2}{\partial x}\right]' D_{12}^{-1}\frac{\partial V_1}{\partial x} + \frac{\varepsilon^2}{4}\left(\frac{\partial V_1}{\partial x}\right)' D_{12}^{-1}D_{22}D_{12}^{-1}\frac{\partial V_1}{\partial x} +\\
&+ x'G_2x = 0,\ V_2(\vartheta,x) = x'C_2x.
\end{aligned} \tag{11}$$

We search the Lyapunov-Bellman functions $V_i(t,x)$ $(i = 1, 2)$ in the kind of the quadratic form

$$V_i(t,x) = x'\Theta_i(t)x,\ \Theta_i'(t) = \Theta_i(t)\ (i = 1,2). \tag{12}$$

Substituting (12) in (11) and taking into account $\frac{\partial V_i(t,x)}{\partial x} = 2\Theta_i x$ $(i = 1, 2)$, we obtain the quadratic forms with respect to the components of the vector x. Equating to zero the coefficients of these quadratic forms, we get the system of two matrix equations

$$\begin{gathered}\dot{\Theta}_1 + \Theta_1(A + D_{21}^{-1}\Theta_2) + (A' + \Theta_2 D_{21}^{-1})\Theta_1 + G_1 + \\ + \Theta_2 D_{21}^{-1} D_{11} D_{21}^{-1}\Theta_2 + \varepsilon^2\Psi_1(\Theta_1,\Theta_2) = 0_{n\times n},\ \Theta_1(\vartheta) = C_1, \\ \dot{\Theta}_2 + \Theta_2 A + A'\Theta_2 + G_2 + \Theta_2 D_{21}^{-1}\Theta_2 + \\ + \varepsilon^2\Psi_2(\Theta_1,\Theta_2) = 0_{n\times n},\ \Theta_2(\vartheta) = C_2,\end{gathered} \tag{13}$$

where by symbol $\Psi_i(\Theta_1, \Theta_2)$ the addends are denoted. They are quadratic with respect to the elements of the matrices Θ_1 and Θ_2.

Let's prove that the system (13) has the extendable to interval $[0, \vartheta]$ solution $(\Theta_1(t), \Theta_2(t))$ for sufficiently small ε.

Indeed, for $\varepsilon = 0$ from (13) we get

$$\begin{gathered}\dot{\Theta}_1 + \Theta_1(A + D_{21}^{-1}\Theta_2) + (A' + \Theta_2 D_{21}^{-1})\Theta_1 + G_1 + \\ + \Theta_2 D_{21}^{-1} D_{11} D_{21}^{-1}\Theta_2 = 0_{n\times n},\ \Theta_1(\vartheta) = C_1,\end{gathered} \tag{14}$$

$$\dot{\Theta}_2 + \Theta_2 A + A'\Theta_2 + G_2 + \Theta_2 D_{21}^{-1}\Theta_2 = 0_{n\times n},\ \Theta_2(\vartheta) = C_2. \tag{15}$$

Now we note that since $G_2 \le 0$, $D_{21} > 0$ and $C_2 \le 0$ then the matrix equation (15) has the unique extendable to interval $[0, \vartheta]$ continuous solution $\Theta_2(t)$ [28, p. 207]. Substituting given $\Theta_2 = \Theta_2(t)$ in (14), we obtain the matrix system of the linear inhomogeneous equations which are relative to Θ_1 with the continuous (with respect to t) coefficients. This equation also has the unique extendable to interval $[0, \vartheta]$ continuous solution $\Theta_1(t)$.Thus, for $\varepsilon = 0$ the system (13) and hence the system (11) has the extendable to interval $[0, \vartheta]$ solution $(\Theta_1(t), \Theta_2(t))$. According to the theorem about the continuous dependence of the solution on parameter [27, p. 8], it follows that for sufficiently small ε the solution of the system (13) is defined on the same interval. Moreover, by Poincare's theorem about analyticity of the solution by parameter [27, p. 8] this solution can be represented in the form of the uniformly convergent on interval $[0, \vartheta]$ series

$$\Theta_i^*(t,\varepsilon) = \Theta_i(t) + \sum_{k=1}^{\infty} \varepsilon^k \Theta_i^{(k)}(t)\ \ (i = 1,2).$$

This fact gives the possibility to search the solution (13) in the form of a series in terms of powers of ε. In conclusion, we notice that by (10) we obtain

$$u_1^B(t,x,\varepsilon) = u_1^B(t,x,V_2(t,x,\varepsilon) = x'\Theta_2(t,\varepsilon)x) = \\ = D_{21}^{-1}\Theta_2(t,\varepsilon)x,\ U_1^B \div u_1^B(t,x,\varepsilon);$$

$$u_2^B(t,x,\varepsilon) = u_2^B(t,x,V_1(t,x,\varepsilon) = x'\Theta_1(t,\varepsilon)x) = \\ = \varepsilon D_{12}^{-1}\Theta_1(t,\varepsilon)x,\ U_2^B \div u_2^B(t,x,\varepsilon).$$

With that the payoffs of the players for any initial positions $(t_0, x_0) \in [0,\vartheta] \times \mathbb{R}^n$ are $\mathscr{J}_i(U^B, t_0, x_0) = x_0'\Theta_i(t,\varepsilon)x_0$ $(i = 1,2)$. This completes the proof.

□

Conclusion

We obtained coefficient criteria for the existence of the Berge-Vaisman equilibrium in a non-cooperative positional linear-quadratic game of two persons with small parameter.

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