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GARANTEED OUTCOMES AND RISKS IN MULTICRITERIA PROBLEM

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Garanteed Outcomes and Risks in Multicriteria Problem.

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Abstract. The method of strategy construction in multicriteria problems under uncertainty (MPU) is examined. Moreover, this strategy provides the Pareto maximality of guaranteed outcome with minimal risk. As application two variants of the problem of diversifying contribution under two deposits (local and foreign) are examined.

Keywords: *Pareto maximum, strategy, uncertainty, vector guarantee, Savage risk, principle of minimax regret*

Introduction

In 1939, Romanian mathematician, who migrated to America in 1938, Abraham Wald (1902–1950) introduced [1] the maximin principle (principle of guaranteed outcome), which allows finding a guaranteed result in a one-criterion problem under uncertainty (OCPU). In almost 10 years, German mathematician Jürg Niehans in 1948 and American mathematician, economist, and statistician Leonard Savage (1917–1971) in 1951 suggested [2, 3] the principle of minimax regret, which allows building guaranteed risks for OCPUs; it would later be referred to as “Savage risk” (later named “Niehans–Savage criterion”). Naturally arises the question of building a strategy, which would simultaneously guarantee a maximum possible outcome and a minimum possible risk.

If we consider the base criterion and the “minus” Savage risk as a second criterion, then the OCPU transforms to two-criteria problem under uncertainty. The present article seeks to provide a mathematical justification of building strategies in multicriteria problems under uncertainty, which would simultaneously increase the guarantees of all outcomes and decrease the accompanying risks.

1. Definition of the problem

A multicriteria problem under uncertainty (MPU) is defined by an ordered set as follows:

$$< X, Y, f_i(x, y)_{i \in \mathbb{N}} > \tag{1}$$

Possible strategies $x \in X \subseteq \mathbb{R}^n$ (belong to the set X from Euclidian n-dimensional space $\mathbb{R}^n$); $Y \subseteq \mathbb{R}^m$ is a subset of pure uncertainties y; criteria $f_i(x, y)$ are scalar functions, defined in the Cartesian product $X \times Y$; $\mathbb{N} = 1, \ldots, N$, and $N \geq 2$ is a set of index numbers of the criteria. In the problem (1), the decision maker seeks to adopt a strategy $x \in X$ to increase values of each criterion $f_i(x, y)$ (called *outcomes*) to maximum possible ones. The decision maker also must take in account the effects from any possible uncertainty $y \in Y$. Ergo each strategy $x \in X$ and each criterion $f_i(x, y)$ is bound to a set of outcomes $f_i(x, Y) = \cup_{y \in Y} f_i(x, y)$. To "narrow down" this set, we will use the Savage risks $R_i(x, y)$, accompanying realizations of each criterion $f_i(x, y)$. At this point, we will proceed to building the Savage *risk function* $R_i(x, y)$ for each criterion $f_i(x, y)$. Under the Nyquist–Savage criterion [2, 3]:

$$R_i(x, y) = \max_{z \in X} f_i(z, y) - f_i(x, y) \qquad i \in \mathbb{N}. \tag{2}$$

Values of $R_i(x, y)$ are called *Savage risks* under the realization of pair $(x, y) \in X \times Y$. Building $R_i(x, y)$ is conducted in two stages: *first*, the function

$$\phi_i[y] = \max_{z \in X} f_i(z, y) \qquad \forall y \in Y \tag{3}$$

is found; *second*, the Savage risk function $R_i(x, y)$ is built under formula (2).

Lemma 1. [4, p. 54] *If sets X and Y are compact and criteria $f_i(x, y)$ are continuous in product $X \times Y$, then*
a) function $\phi_i[y]$ from (3) *is continuous in Y;*
b) functions

$$f_i[x] = \min_{y \in Y} f_i(x, y) \qquad \forall x \in X \tag{4}$$

are continuous in X.

Remark 1. From (4) follows that $f_i[x]$ is a guarantee for $f_i(x, y)$ for any $y \in Y$, since, according to (4),

$$f_i[x] \leq f_i(x, y) \qquad \forall y \in Y, \; i \in \mathbb{N}.$$

From this point, *comp* $\mathbb{R}^n$ is the designation for the set of compacts within $\mathbb{R}^n$, and the continuousness of $f_i(x,y)$ in $X \times Y$ is denoted as $f_i(x,y) \in C(X \times Y)$.

Remark 2. In problem (1), if criterion $f_i(x,y) \in C(X \times Y)$ and $X \in comp\,\mathbb{R}^n$, $Y \in comp\,\mathbb{R}^n$, then the Savage risk function $R_i(x,y)$ $(i \in \mathbb{N})$ in (2) is continuous in $X \times Y$. Indeed, continuousness of $\phi_i[y]$ in (3) follows from Lemma 1, statement a), and, accordingly, the Savage risk function is continuous as a difference of continuous functions: $R_i(x,y) = \phi_i[y] - f_i(x,y)$ $(i \in \mathbb{N})$.

Remark 3. The Savage risk function from (2) characterizes deviation of criterion $f_i(x,y)$ from the "desired" value $\max_{x\in X} f_i(x,y)$. This prompts the decision maker to choose a strategy $x \in X$, which would decrease the difference $R_i(x,y)$ from (2) to the greatest extent possible or, equivalently, increase $-R_i(x,y)$. Then the given MPU (1) is assigned to a $2N$-criteria problem

$$\left\langle X, Y, f_i(x,y), -R_i(x,y)_{i\in\mathbb{N}} \right\rangle. \tag{5}$$

The goal of the decision maker in (5) is to choose a strategy $x \in X$ so that the $2N$ criteria $f_i(x,y), -R_i(x,y)$ $(i \in \mathbb{N})$ assume the greatest values possible; the decision maker must also expect a possibility of emergence of uncertainty $y \in Y$.

2. Formalization of a guaranteed solution

MPUs are well-described in the literature (in particular, we refer to the monography [5] published in the United States). In the present article, we will use the approaches from a recent series of articles [6, 7]. In this formalization, we will use two N-vectors, $f = (f_1, \ldots, f_N), R = (R_1, \ldots, R_N)$, and, finally, the $2N$-vector $F = (F_1, \ldots, F_N, F_{N+1}, \ldots, F_{2N})$, where $F_i = f_i, F_{i+N} = -R_i$ $(i \in \mathbb{N})$. Then (5) may be rewritten as

$$\left\langle X, Y, \{F_j(x,y)\}_{j=1,\ldots,2N} \right\rangle. \tag{6}$$

Remark 4. Consider the guarantees of the risk functions $-R_i(x,y)$ $(i \in \mathbb{N})$:

$$-R_i[x] = \min_{y\in Y}[-R_i(x,y)] = \min_{y\in Y}[-F_{N+i}(x,y)] = -\max_{y\in Y} F_{N+i}(x,y) = -F_{N+i}[x],$$

from which follows the equation:

$$R_i[x] = \max_{y\in Y} R_i(x,y)(i \in \mathbb{N}) \tag{7}$$

and thus $R_i[x] \geq R_i(x,y)$ $\forall y \in Y$. Then $R_i[x]$ is the guarantee of Savage risk function for the decision maker using strategy $x \in X$ (it must be remembered that that decision maker strives to decrease the risk and, according to (7), the risk function $R_i(x,y)$ cannot exceed $R_i[x]$ for any $y \in Y$).

We will now proceed to building a *multicriteria problem of guarantees*. For the problem (1) and, accordingly, in the MPUs extended with risks (5), (6), we will only limit ourselves to *interval* uncertainties $y \in Y$. We only know that y may assume any value from Y, and probabilistic characteristics of arrangement of $y \in Y$ are absent or unknown for some reasons. Thus, the MPU (5) (or, equivalently, (6)) is assigned to a *multicriteria problem of guarantees*

$$\left\langle X, \left\{ F_j[x] = \min_{y \in Y} F_j(x,y) \right\}_{j=1,\ldots,2N} \right\rangle. \tag{8}$$

We recall that $F_i[x] = \min_{y\in Y} f_i(x,y)$ and $F_{i+N}[x] = -\max_{y\in Y} R_i(x,y) = R_i[x]$ $(i \in \mathbb{N})$. For a formalization of an optimal solution of (1) under risks we will use the term, introduced in 1909 [8] by Italian economist and sociologist Vilfredo Pareto (1848–1923), later named “the Pareto maximum”.

Definition 1. Strategy $x^P \in X$ is called **Pareto maximal** (sometimes also **effective**) for a $2N$-criteria problem (8), if for any $x \in X$, the system of inequalities

$$F_j[x] \geq F_j[x^P] \qquad (j = 1, \ldots, 2N),$$

of which at least one inequality is strict, is not consistent.

Lemma 2. [9, p. 71] *Assume* $\alpha_j = const > 0$ $(j = 1, \ldots, 2N)$ *, then* $x^P \in X$*, complying with the equation*

$$\sum_{j=1}^{2N} \alpha_j F_j[x^P] = \max_{x \in X} \sum_{j=1}^{2N} \alpha_j F_j[x],$$

is the Pareto maximal strategy for (8)*.*

Corollary 1. *If* $X \in comp\ \mathbb{R}^n$ *and* $F_j[x] \in C(X)$ $(j = 1, \ldots, 2N)$*, then a Pareto maximal strategy exists for* (8)*.*

Definition 2. A tripod $(x^P, f[x^P], R[x^P]) \in X \times \mathbb{R}^{2N}$ is called **Pareto guaranteed under outcomes and risks solution of an MPU** (1), if:

1) the following functions exist: $f_i[x] = \min_{y\in Y} f_i(x,y)$, $R_i[x] = \max_{y\in Y} R_i(x,y)$ $\forall x \in X$ $(i \in \mathbb{N})$,

2) x^P is Pareto maximal for a multicriteria problem of guarantees (8).

It must be remembered that

$$f[x] = (f_1[x], \dots, f_N[x]),\ R[x] = (R_1[x], \dots, R_N[x]),$$
$$R_i[x] = \max_{y \in Y} R_i(x, y),\ R_i(x, y) = \max_{z \in X} f_i(z, y) - f_i(x, y).$$

Why is the Pareto guaranteed under outcomes and risks solution (PGOR) suggested as a "good" solution of the MPU (1)?

First, it provides an answer to the indigenous Russian question: "What is to be done?" It is suggested the decision maker follow the strategy x^P of the tripod $(x^P, f[x^P], R[x^P])$.

Second, this strategy x^P "provide" outcomes $f_i(x^P, y)$ greater than or equal to $f_i[x^P]$ with a risk of $R_i(x^P, y)$ not greater than $R_i[x^P]$ for any $i \in \mathbb{N}$, under implementation of any uncertainty $y \in Y$ (i.e. x^P set lower limits for outcomes implemented under $x = x^P$ and upper limits for risks accompanying this implementation).

Third, situation x^P implements "the greatest" Pareto maximal outcomes and corresponding "minus" risks; otherwise speaking, there is no situation $x \neq x^P$, in which all risk guarantees $f_i[x^P]$ would increase and at the same time, all risk guarantees would decrease $R_i[x^P]$.

It should be noted that the union of "second" and "third" is somewhat analogous to the maximin strategy for a one-criterion problem under uncertainty, with the difference being the substitution of the inner minimum with $min_{y \in Y} F_i(x, y)$ $(i \in \mathbb{N})$ and the substituition of the outer maximum with the Pareto maximum. Two possible research routes are available at this point. The first one of them is to substitute the Pareto maximum with the Slater, Borwein, Geoffrion optimums and conical optimality, and to establish connection between such different solutions. The second route is based on the desire of the decision maker for higher profits, while the guarantees in Definition 2 are the "lowest". Therefore it is possible to substitute scalar minimums (from the inner minimum in the maximin) with a vectorial one (listed above), therefore increasing the guarantees. Connections between solutions are also of interest (some attempts to build such connection the first author lists in the monography [5]).

Fourth, Definition 2 allows suggesting a constructive method of building a PGOR. It may be reduced to four steps.

Step 1. Using $f_i(x, y)$, find $\max_{x \in X} f_i(x, y) = \phi_i[y]$ and build the Savage risk function for criterion $f_i(x, y)$, assuming $R_i(x, y) = \phi_i[y] - f_i(x, y) (i \in \mathbb{N})$.

Step 2. Build outcome guarantees $f_i[x] = \min_{y \in Y} f_i(x, y)$ and risk guarantees $R_i[x] = \max_{y \in Y} R_i(x, y)$ $(i \in \mathbb{N})$.

Step 3. For a $2N$-criteria guarantees problem $\langle X, Y, \{f_i[x], -R_i[x]\}_{i\in\mathbb{N}}\rangle$, calculate a Pareto maximal stategy x^P. Lemma 2 may be used here, assuming $\alpha_j = 1$ $(j = 1, \dots, 2N)$.

Step 4. Using x^P, determine the values of guarantees $f_i[x^P]$ and $R_i[x^P]$ $(i \in \mathbb{N})$ and collect them into two N-vectors $f[x^P] = (f_1[x^P], \dots, f_N[x^P]), R[x^P] = (R_1[x^P], \dots, R_N[x^P])$.

The resulting tripod $(x^P, f[x^P], R[x^P])$ forms the desired PGOR, which complies with Definition 2, i.e. using strategy x^P results in a guaranteed outcome $f_i[x^P]$ with a guaranteed Savage risk $R_i[x^P]$ for each criterion $f_i(x, y)$ $(i \in \mathbb{N})$.

We will now proceed to the proof of existence of PGOR for requirements "usual" for mathematical programming.

Theorem 1. *If in problem* (1),

1. *sets X and Y are compact,*
2. *criteria $f_i(x, y)$ are continuous in $X \times Y$, then a PGOR exists.*

Proof. Given the continuousness of the criteria $f_i(x, y)$, according to Lemma 1, functions $f_i[x] = \min_{y\in Y} f_i(x, y)$, $\phi_i[y] = \max_{x\in X} f_i(x, y)$ $(i \in \mathbb{N})$ are continuous; according to Remark 2, this implies that the Savage risk functions $R_i(x, y) \in C(X \times Y)$ are continuous as well. From Lemma 1 also follows the continuousness of the functions $R_i[x] = \max_{y\in Y} R_i(x, y)(i \in \mathbb{N})$. So, under compliance with both requirements of the theorem, in a multicrireria problem of guarantees (8) all criteria $F_j[x]$ are continious $(j = 1, \dots, 2N)$. As follows from Corollary 1, a Pareto maximal strategy x^P exists in (8), and for this strategy, vectors $f[x^P]$ and $R[x^P]$ are immediately defined. The tripod $(x^P, f[x^P], R[x^P])$ forms the sought-after PGOR (8). ☐

3. Risks and outcomes for diversification of a deposit into sub-deposits in different currencies

In publications on microeconomics, for example, in [10, p. 103] all decision makers are divided into three categories: risk-averse, risk-neutral, and risk-seeking. In the present article, a solution of the problem of diversification of a deposit into sub-deposits in rubles and in a foreign currency is found for a risk-neutral person in two cases. It should be noted that another article [11, p. 9] addresses the problem, and it presents results different from those below. The case is that Pareto solutions form a set the elements of which are in general different. Different elements of the same Pareto set appear in [11] as well as in the present article. We will now proceed to the diversification problem.

The amount of money in a singular deposit diversified into two sub-deposits (in rubles and a foreign currency) accumulated by the end of the year may [12, p. 58–60] be represented as

$$f(x, y) = x(1 + r) + \frac{1 - x}{k}(1 + d)y \tag{9}$$

where r and d are the rates of interest for the sub-deposits in rubles and a foreign currency, accordingly; k and y are the exchange rates (to the ruble) in the beginning and the end of the year, accordingly; $x \in [0, 1]$ is a fraction defining the proportion, in which the singular deposit is divided into the sub-deposits. In accordance with (9), x is then the fraction of the sub-deposit in rubles, and the remaining part $1-x$ is converted into a foreign currency $\frac{1-x}{k}$ and allocated in a sub-deposit. In the end of the year it is converted back into rubles $\frac{1-x}{k}(1 + d)y$ and the resulting amount of money is determined by the sum in (9). It is required to determine the fraction x, under the implementation of which the resulting amount of money $f(x, y)$ is the greatest possible one. It must be taken in account the future exchange rate y is normally unknown. We will, however, assume it may be defined in a corridor of possible values, precisely, $y \in [a, b]$, where the constants $b > a > 0$ are set or obtained at first. So, the mathematical model of the diversification problem in hand may be represented as an ordered tripod

$$\Gamma = \langle X = [0, 1], Y = [a, b], f(x, y)\rangle\,, \tag{10}$$

where function $f(x, y)$ is defined in (9); here, $X = [0, 1]$ is the set of strategies x of the decision maker; $Y = [a, b]$ is the set of uncertainties y; finally, $f(x, y)$ is the function of usefulness (criterion), the value of which is called outcome. From the perspective of operations research, Γ is a one-criterion problem of making decisions under uncertainties. Presence of an uncertainty, as well as the intent to consider it is strongly tied to risks — "the possibility of deviation of any variables from the desired values" [13, p.15].

Statement. *For the diversification problem of a deposit into sub-deposits in rubles and a foreign currency, the Pareto guaranteed under outcomes and risks solution is*

$$(x^P, f[x^P], R[x^P]) = \begin{cases} \left(0, \dfrac{1+d}{k}a, 0\right), & k\dfrac{1+r}{1+d} \leq a, \\ (1, 1+r, 0)\,, & k\dfrac{1+r}{1+d} \geq b. \end{cases}$$

Proof. Consider two cases:
first, $k\frac{1+r}{1+d} \leq a$,
second, $k\frac{1+r}{1+d} \geq b$.

These cases correspond to the two possibilities of relative of the point $k\frac{1+r}{1+d}$ and the interval $[a, b]$ in the y-axis (see Fig. 1).

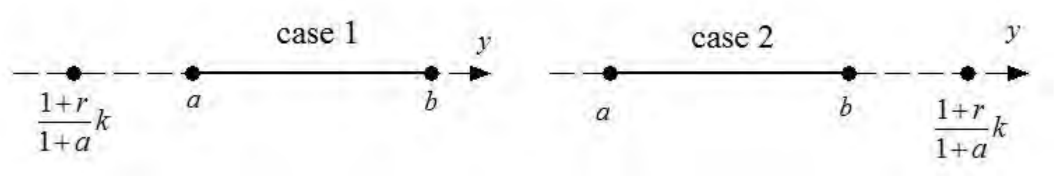

Fig. 1. Possible cases of relative location of the interval $[a, b]$ and the point $k\frac{1+r}{1+d}$.

Case 1. $\left[k\frac{1+r}{1+d} \le a\right] \implies \left[1 + r \le \frac{1+d}{k}a\right]$. Using the suggested method of finding a PGOR (after Definition 2), we get the following results:

Stage 1. Building the Savage risk functions in accordance with (2). We have to find the x-maximal value of the criterion $f(x, y)$, i.e. $\max_{x\in X} f(x, y)$. For that, we will make an upper estimate of this function and show that it is attainable (and thus the optimal one). Indeed,

$$f(x,y) = x(1+r) + \frac{1-x}{k}(1+d)y \le \frac{1+d}{k}ax + \frac{1-x}{k}(1+d)y =$$
$$= \frac{1+d}{k}x\underbrace{(a-y)}_{\le 0} + \frac{1+d}{k}y \le \frac{1+d}{k}y.$$

And then the maximum is implemented under $x = 0$. Therefore, $\max_{x\in X} f(x, y) = \frac{1+d}{k}y$. Then the risk function $R(x, y) = \frac{1+d}{k}y - x(1 + r) - \frac{1-x}{k}(1 + d)y = x\frac{1+d}{k}(y - \frac{1+r}{1+d}k)$.

Stage 2. Building guarantees of outcomes and risks. It is easy to see that

$$f[x] = \min_{a\le y\le b} f(x,y) = f(x,a) = x\frac{1+d}{k}\left(\frac{1+r}{1+d}k - a\right) + \frac{1+d}{k}a,$$
$$-R[x] = \min_{a\le y\le b}[-R(x,y)] = -R(x,b) = -x\frac{1+d}{k}\underbrace{\left(b - \frac{1+r}{1+d}k\right)}_{\ge 0}.$$

Stage 3. Finding the Pareto maximal strategy x^P in the two-criteria problem $< X, f[x], -R[x] >$. For that, we may find x^P knowing that

$$\max_{x\in[0,1]}(f[x] - R[x]) = f[x^P] - R[x^P].$$

Consider the case 1 (see Fig. 2), where the bold line denotes the graph of the sum of the functions.

Stage 4. From the figure 2 follows that $\max_{x\in[0,1]}(f[x]-R[x])$ is achieved in the point $x=0$. Thus, $x^P=0$. Therefore, $f[x^P]=\frac{1+d}{k}a$ and $R[x^P]=0$.

Case 2. Analogous reasoning may be provided for case 2. The implication $\left[k\frac{1+r}{1+d}\geq b\right] \implies \left[1+r\geq \frac{1+d}{k}b\right]$ is obvious.

Stage 1. The maximum of the function

$$f(x,y)=x(1+r)+\frac{1-x}{k}(1+d)y\leq x(1+r)+(1-x)\underbrace{\frac{1+d}{k}b}_{<1+r}\leq$$
$$\leq x(1+r)+(1-x)(1+r)=1+r$$

is attained under $x=1$. Thus, $\max_{x\in X}f(x,y)=1+r$. Therefore, the risk function $R(x,y)=(1+r)-x(1+r)-\frac{1-x}{k}(1+d)y=(1-x)\frac{1+d}{k}(\frac{1+r}{1+d}k-y)$.

Stage 2. Finding the guarantees of outcomes and risks.

$$f[x]=\min_{a\leq y\leq b}f(x,y)=f(x,a)=x\frac{1+d}{k}\left(\frac{1+r}{1+d}k-a\right)+\frac{1+d}{k}a,$$
$$-R[x]=\min_{a\leq y\leq b}[-R(x,y)]=-R(x,a)=-(1-x)\frac{1+d}{k}\underbrace{\left(\frac{1+r}{1+d}k-a\right)}_{\geq 0}.$$

Stage 3. The requisite graphs are found in Fig. 3, with the bold line representing the sum of the two functions $f[x]-R[x]$.

Stage 4. Therefore, $\max_{x\in[0,1]}(f[x]-R[x])$ is attained in the point $x=1$. Thus, $x^P=1$ and, accordingly, $f[x^P]=1+r$ and $R[x^P]=0$.

Therefore, a Pareto guaranteed solution under outcomes and risks for the problem of diversification of a deposit into two sub-deposits, one in rubles and one in a foreign currency, is

$$(x^P,f[x^P],R[x^P])=\begin{cases}\left(0,\dfrac{1+d}{k}a,0\right), & k\dfrac{1+r}{1+d}\leq a,\\[2ex] (1,1+r,0), & k\dfrac{1+r}{1+d}\geq b.\end{cases}$$

□

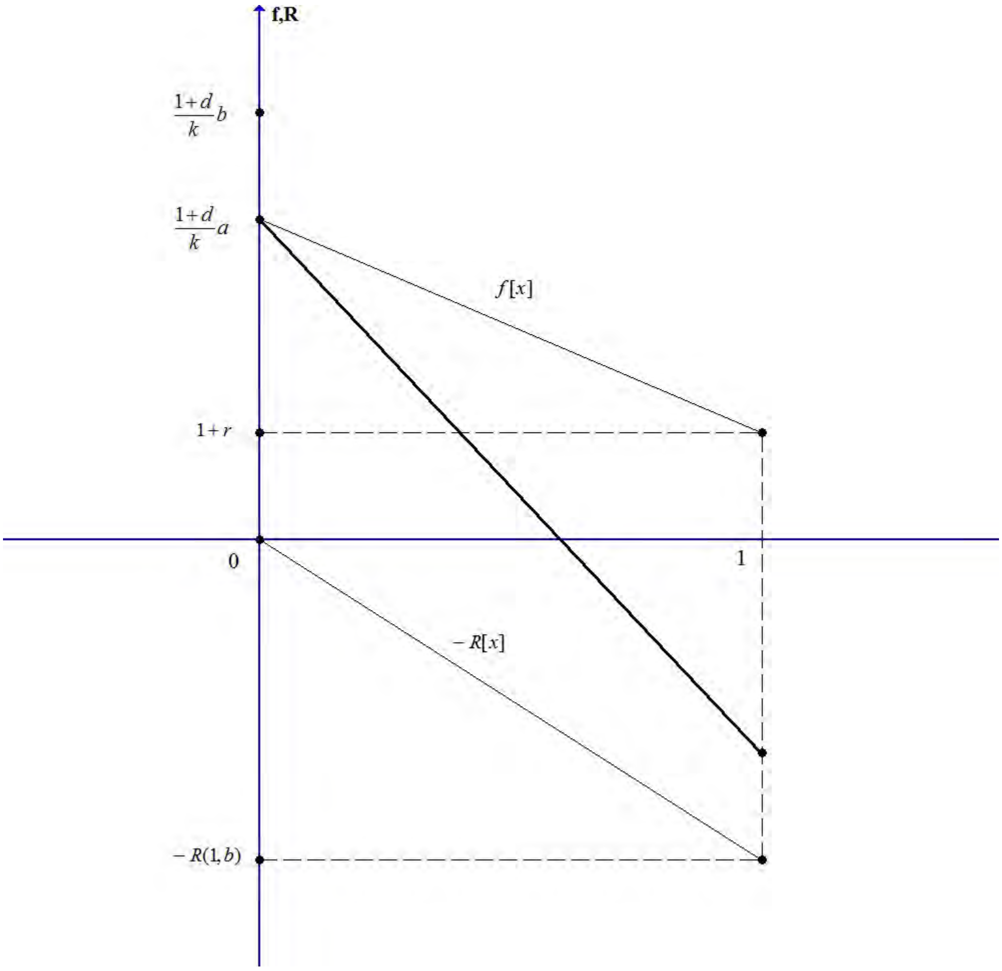

Fig. 2. Pareto maximal strategy for case $k\frac{1+r}{1+d} \leq a$.

Conclusion

A solution of the problem of diversification of a deposit into sub-deposits in rubles and in a foreign currency is found for a risk-neutral person in two cases.

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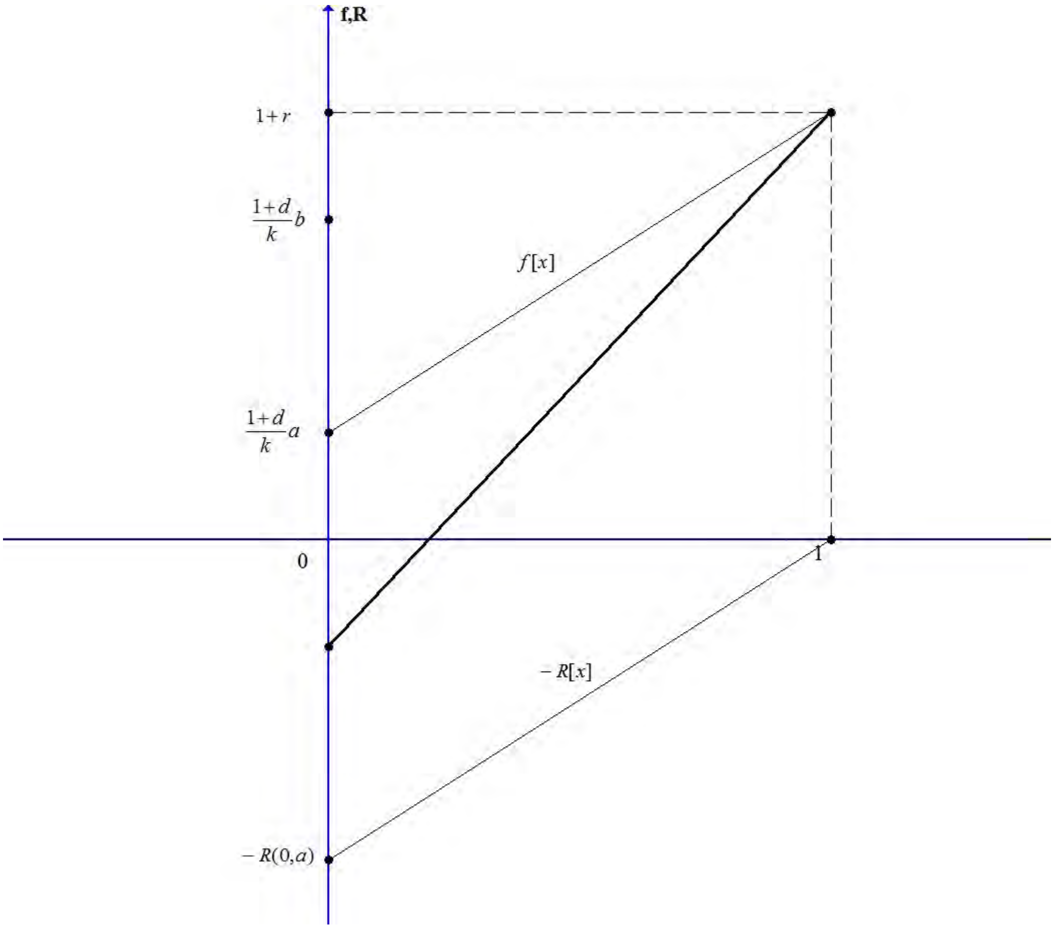

Fig. 3. Pareto maximal strategy for case $k\frac{1+r}{1+d} \geq b$.

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